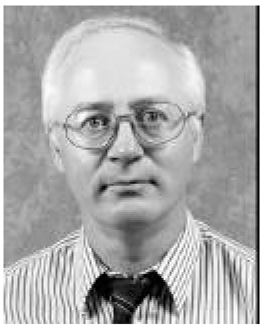
A Nonuniform Circular-Motion Experiment

Desmond N. Penny



Desmond Penny is a Professor of Physics at Southern Utah University where he was recently honored as a Distinguished Educator. He received his B.Sc. and M.Sc. degrees in Mathematical Physics from University College Cork, Ireland, and his Ph.D. from the University of Utah. He was appointed an Accredited Mathematica Trainer by Wolfram Research in 1999 and enjoys using Mathematica for the analysis of geometry and physics problems.

Southern Utah University
Cedar City, UT 84720;
penny@suu.edu

I recently developed an experiment that uses one of the most useful and general equations of physics, the work-energy equation, to analyze rotational motion under gravity. In this paper, I show two methods to develop the theory for this experiment. The first requires a mathematics level that, with some guidance, is within the capability of students who have had one or two quarters of calculus. This method most accurately models the experimental data. The second is an approximate method that involves only a straightforward application of the work-energy equation and simple algebra.

The central problem of the lab experiment is to develop an equation for the period of a mass that rotates at the end of a string in a vertical circle where the tension in the string at the top of the circle is zero.

Experimental Procedure

The experiment requires the following items: string, a small mass, a meter stick, an alligator clip commonly used in electric circuits, and a stopwatch. The procedure for the experiment is as follows:

1. Tie the string to the mass and measure the length of the string to the center of mass. It is convenient to mark the string at the distances 0.2, 0.3, 0.4,. . . , 1.9, 2.0 m.
2. In order to reduce the frictional effect of the string rubbing against your finger, fasten the alligator clip at the desired mark on the string. This procedure gives a better defined radius for the mass and also reduces the friction, giving a more consistent speed of rotation. The use of the clip dramatically improves the accuracy of the experiment.
3. Rotate the mass in a vertical circle. With practice you can feel the definite reduction in tension at the top of the circle. (We've found that using a set of several keys as the mass can work better than an ordinary pendulum bob. They provide an audible click as the tension goes to zero at the top of the path.)
4. Time the period of rotation.

Data

My students determined the total time for 20 revolutions and then divided by 20 to find the period. Five trials were made for each radius and the average found. These average values are shown in Table I. In order to ensure adequate coverage of a wide range of radii, we had each group of students take data for only one specific radius. The table forms the basis for the graph shown in Fig. 3, which is a plot of the average period, T, versus R taken from

Table I. Measurement of Period *T*, for each radius, *R*.

R (m)	Times for 20 Rotations (s) Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Average of Times (s)	T (s)
0.2	12.0	11.35	11.42	11.46	11.34	11.51	0.5757
0.3	14.04	15.17	15.20	13.91	14.29	14.52	0.7261
0.4	16.66	15.85	17.05	16.94	16.50	16.60	0.8300
0.5	18.20	18.32	18.54	18.18	18.63	18.37	0.9187

columns 8 and 1, respectively, in Table I. The error bars in Fig. 3 give the maximum and minimum periods recorded for that radius.

Plotting the data can be conveniently done with any spreadsheet program. If the program does not easily handle error bars, these can be later superimposed by hand for each point. For this paper, I have used *Mathematica*[1] to plot the data shown in Fig. 3. I chose *Mathematica* primarily because it can conveniently display error bars on each datum point.

There are a number of possible sources of error in the experiment described. One is the difficulty in keeping the rotation rate consistent. The effect of this error is difficult to quantify. However, I have found with sufficient practice, one can feel the tension in the string slacken at the top of the loop. Another source of error is due to the slight motion of the center of the circle due to small movements of the experimenter's hand. Finally, there is the uncertainty involved in timing the period. The good reproducibility of the time measurements shown in Table I and also the excellent agreement of the measured periods with the values predicted by the model described in the next section suggest that the combination of these errors is very small.

Calculus-Based Model

There are three stages in the analysis. The first is to find an expression for the speed of the mass at the top of the circle when the tension in the string is zero. The second involves using the work-energy equation to find an expression for the speed of the mass at any other point in the motion. The third and final stage is to use this expression to find the period of the rotation. Only the last stage requires calculus-level math.

Stage 1: Finding Speed at Top of Circle

Consider the free body diagram of the mass at the top of the circle, Fig. 1.

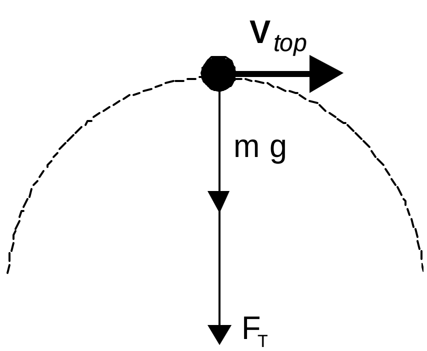


Fig. 1. Free-body diagram of mass at top of circle.

Applying Newton's second law gives:

$$F_t + mg = m\frac{v^2_{top}}{R}\ , \qquad (1)$$

where F_t is the tension in the string and v_{top} is the speed of the mass, m, at the top of the circle of radius R. In this experiment, we assume that F_t 0 at the top of the circle, and so we can solve Eq. (1) for v_{top} to give:

$$v_{top} = \sqrt{Rg}. \qquad (2)$$

Stage 2: Finding Speed at Any Point on Circle

Regarding Fig. 2., take the initial point, P_i, of the motion at the top of the circle and consider any other point of the motion, P_f. R is the radius of the circle and θ is the subtended angle in radians.

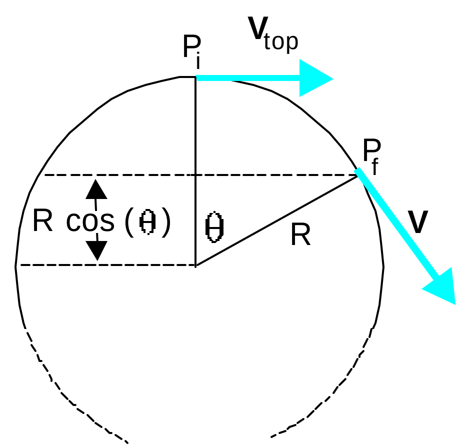


Fig. 2. Geometry of motion from P_i to P_f.

At point P_i, $v = v_{top}$. We want to find an expression for the speed at the point P_f. The only forces acting on the mass between P_i and P_f are the tension F_t, the weight of the body, and the frictional force of the air. If $v = v_{top}$ at the peak of each orbit, then the negative work done by friction during each orbit must be balanced by an equal amount of positive work done by the experimenter. This work is done near the bottom of each revolution as the experimenter delivers a slight upward impulse to increase the energy of the mass. Since gravity is a conservative force, the work it does may be calculated from the change in the potential energy:

$$\Delta PE = mg\Delta y = mg(R\cos(\theta) - \mathrm{R}) = mgR(\cos(\theta) - 1). \qquad (3)$$

Note that in Eq. (3), Δy is negative because the body moves downward from P_i to P_f. The change in kinetic energy is

$$\Delta K = K_f - K_i = \tfrac{1}{2}m(v^2 - v^2_{top}). \qquad (4)$$

Applying the work-energy theorem

$$0 = \Delta PE + \Delta KE \qquad (5)$$

gives

$$0 = ½m(v^2 - v^2_{top}) + mgR(\cos(\theta) - 1). \quad (6)$$

Substituting for v_{top} from Eq. (2) and solving for v gives

$$v = \sqrt{2gR[1-\cos(\theta)] + gR} = \sqrt{gR}\sqrt{3 - 2\cos(\theta)}. \quad (7)$$

The efforts of friction and the small motions of the experimenter's hand (discussed above) will cause the actual values of v to differ a bit from those given by Eq. (7). The actual velocity will be slightly larger for half of the orbit (as the mass moves upward) and slightly smaller for the other half. We expect a cancellation of these effects when we integrate over a complete orbit to find the period.

Stage 3: Finding Period of Motion

This part of the analysis requires integral calculus. A little guidance might be required here for some groups, depending on the math background of the students. Refer to Fig. 2. If s is the arc length from P_i to P_f, then

$$s = R\theta. \quad (8)$$

In differential form, for fixed R, we have

$$ds = Rd\theta, \quad (9)$$

which gives

$$\frac{ds}{dt}dt = vdt = Rd\theta. \quad (10)$$

Solving for dt, we have

$$dt = \frac{R}{v}d\theta. \quad (11)$$

Substituting for v from Eq. (6) gives

$$dt = \frac{Rd\theta}{\sqrt{gR}\sqrt{3-2\cos\theta}}. \quad (12)$$

In order to find the period, T, of the motion, integrate both sides of this equation through one complete rotation. The result is

$$T = \sqrt{\frac{R}{g}}\int_0^{2\pi}\frac{d\theta}{\sqrt{3-2\cos(\theta)}}. \quad (13)$$

This integral can be evaluated using a numerical routine such as Simpson's Rule,[2] or a computer program such as *Mathematica*.[1,3] In either case, the result is the same,

$$T = (4.0378)\sqrt{\frac{R}{g}}. \quad (14)$$

We note in passing the resemblance of this equation to that of the period of a simple pendulum

$$T = 2\pi\sqrt{\frac{L}{g}}, \quad (15)$$

where L = length of the pendulum. It is interesting to note that the period of a body executing a complete circle subject to zero tension at the top of the motion is about two-thirds that of a simple pendulum of the same length.

Comparison of Theory and Experiment

Now comes the test of our theory—how does it compare with our experimental results? Using the value of g = 9.8 m/s^2, I superimposed the theoretical Eq. (14) on the experimental data, see Fig. 3. The solid line in the figure is the theoretical curve Eq. (14). The data points are represented by small squares and the error bars are also shown. The average period for each radius R is plotted. The error bars show the maximum and minimum period recorded for each radius.

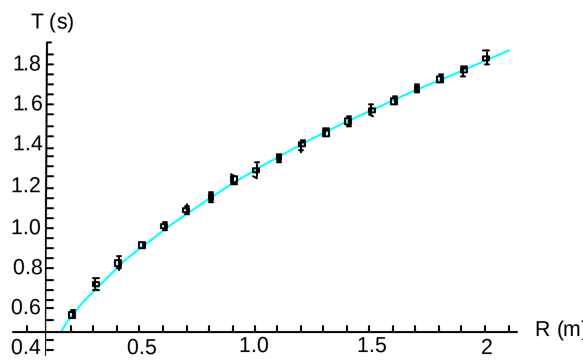


Fig. 3. Comparison of theory with experiment.

Algebra-Based (Approximate) Model

For students who have not had calculus, it is possible to develop a simple model that does an adequate job of modeling the data. A brainstorming session with the students can usually begin by asking them where is the speed of the body least? Where is the speed greatest? How could we find the period of the motion by analysis? If the students have been exposed to the analysis given in Stage 1, they will quickly realize that the speed is minimum at the top and maximum at the bottom. They will also realize that we can easily solve for these speeds given the radius of the circle. The period of the motion can be found from

$$v_{av} = \frac{2\pi R}{T}, \quad (16)$$

where v_{av} is the average speed of the body. Solving for the period, T, gives

$$T = \frac{2\pi R}{v_{av}} . \tag{17}$$

We can get a rough estimate of the period by finding the average speed using

$$v_{av} = \frac{v_{top} + v_{bot}}{2} , \tag{18}$$

where, as before, v_{top} is the speed of the body at the top and v_{bot} is the speed at the bottom. We can then use Eq. (17) to find the period.

In order to complete the development of this model, it only remains to show the derivation of v_{bot}.

Finding Speed at Bottom of Circle

In Fig. 4, we see the geometry of the motion of the body from the top, P_i, to the bottom, P_f.

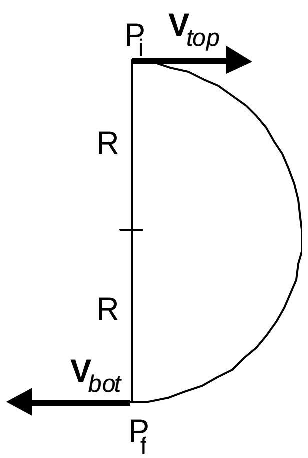


Fig. 4. Geometry of motion from top of circle, P_i, to bottom, P_f.

Using the work-energy analysis as before, we find

$$v_{bot} = \sqrt{5gr} . \tag{19}$$

Substituting Eqs. (19) and (2) into Eq. (17) gives

$$v_{av} = \left(\frac{1+\sqrt{5}}{2}\right)\sqrt{gr} . \tag{20}$$

Substituting this into Eq. (17) gives

$$T = \frac{4\pi}{(1+\sqrt{5})}\sqrt{\frac{R}{g}} = (3.8832)\sqrt{\frac{R}{g}} . \tag{21}$$

In Fig. 5, compare this model with the experimental data.

We note that the theoretical curve from Eq. (21) is significantly lower than the experimental data. While this may be quite acceptable, if better agreement is desired, the model may be improved. This can be done by computing v_{av} using not only v_{top} and v_{bot}, but also the velocities at a number of points in between. The necessary calculations may be done using Eq. (7) and a simple spreadsheet program.

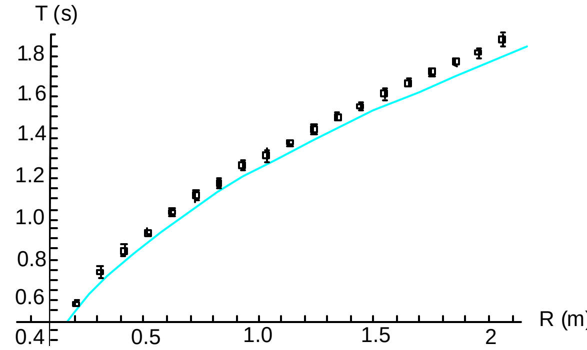


Fig. 5. Comparison of approximate model with experiment.

Conclusion

We have run this experiment for students with calculus- and algebra-based mathematical backgrounds. We find that the lab runs well, giving students an excellent exposure to gathering data, using spreadsheets, deriving a theoretical model, and comparing theory with experiment. The agreement between the calculus-based theory and experimental data is remarkable for the calculus-based model and adequate for the algebra-based model. The students (and instructor) invariably get a real kick out of the fact that they can get such good results even though the equipment used is so simple.

Acknowledgements

The author would like to thank Jake Wempen for his help in the preparation of the figures, and Laura Cotts for running this experiment for multiple groups in her physics lab at Southern Utah University. This resulted in a much smoother final procedure for the experiment. The author would also like to thank her for her critiques of the ideas contained in this paper.

References

1. *Mathematica*, Wolfram Research, Inc., 100 Trade Center Drive, Champaign, IL 61820-7237.
2. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. (Dover Publications, 1970).

(Editor's Note: For those using the Maple programming language by Waterloo, it is necessary to use the evalf (Int ( )) structure to properly evaluate the integral.)